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A Generalization of Poincaré's Theorem for Recurrence Equations*

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A natural generalization to systems of first order equations is given of Poincaré's classical theorem on ratio asymptotics of solutions of higher order recurrence equations. © 1990 Academic Press, Inc.

A key result in the theory of recurrence equations is the following theorem of Poincaré (see [6, Sect. 2, pp. 213–217, and Sect. 6, p. 237], or, for secondary sources, [1, p. 391; 4, Sect. 17.1, p. 526; 5, Sect. X.6, p. 300]).

THEOREM 1. *Let k be a positive integer. Suppose that for every integer $n>0$ the difference equation*

$$f(n+k)+\sum_{j=0}^{k-1} a_{jn} f(n+j)=0 \tag{1}$$

holds, where the limits

$$\lim_{n\to\infty} a_{jn}=a_j \qquad (0\leqslant j<k) \tag{2}$$

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exist, and the roots of the "characteristic equation"

$$z^k + \sum_{j=0}^{k-1} a_j z^j = 0 \tag{3}$$

all have different absolute values. Write $\zeta_1, ..., \zeta_k$ for these roots. Then either $f(n) = 0$ for all large enough n, or there is an l with $1 \leqslant l \leqslant k$ such that

$$\lim_{n \to \infty} f(n+1)/f(n) = \zeta_l. \tag{4}$$

This result has applications, e.g., to the study of the asymptotic behavior of orthogonal polynomials. The example of the Legendre polynomials is mentioned by Poincaré himself (see [6, p. 252]); for a recent discussion of certain applications of Poincaré's theorem to orthogonal polynomials, see, e.g., [3, Sect. 2]. The aim of these notes is to prove the following generalization of the above theorem.

THEOREM 2. *Let k be a positive integer, and let $\mathbf{A}_n$ be a $k \times k$ matrix such that*

$$\lim_{n \to \infty} \mathbf{A}_n = \mathbf{A} \tag{5}$$

exists. Suppose that all eigenvalues of the matrix $\mathbf{A}$ have different absolute values. Write $\mathbf{v}_1, ..., \mathbf{v}_k$ for the (column) eigenvectors of $\mathbf{A}$. Let the k-dimensional column vectors $\mathbf{u}_n$ be such that

$$\mathbf{u}_{n+1} = \mathbf{A}_n \mathbf{u}_n \qquad (n > 0). \tag{6}$$

Then either $\mathbf{u}_n = 0$ for all large enough n, or $\mathbf{u}_n \neq 0$ for all large enough n, and in this case there is an l with $1 \leqslant l \leqslant k$ and a sequence of complex numbers c_n such that

$$\lim_{n \to \infty} c_n \mathbf{u}_n = \mathbf{v}_l. \tag{7}$$

This result arises naturally if one restates the higher order equation given in (1) as a system of first order equations. However, what one obtains in this way is a real generalization; that is, there seems to be no way of making use of the statement of Theorem 1 in the proof of Theorem 2. Nonetheless, the proof of Theorem 2 we are about to present does make use of ideas contained in the standard proof (which is essentially Poincaré's original proof), given in the cited sources, of Theorem 1. (However, all of these sources give the proof only in case $k = 3$ except for [1]; this book gives the proof for an arbitrary k, but the presentation is much too

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complicated.) First we are going to give the proof of Theorem 2, and then we will show how Theorem 1 follows from Theorem 2.

Proof. If $\mathbf{u}_n = 0$ for some n, then $\mathbf{u}_{n'} = 0$ for all $n' \geqslant n$. The conclusion of the theorem holds in this case; so assume that $\mathbf{u}_n \neq 0$ for any positive integer n. Write $\mathbf{u}_n$ as a linear combination of the eigenvectors of $\mathbf{A}$:

$$\mathbf{u}_n = \sum_{j=1}^{k} \rho_{jn} \mathbf{v}_j. \tag{8}$$

Write λ_j for the eigenvalue corresponding to the eigenvector $\mathbf{v}_j$; assume that these eigenvalues are arranged in the order of decreasing absolute value:

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_k|.$$

Let $\|\cdot\|$ denote the l^2 norm on the space of k-dimensional column vectors. Without loss of generality, we may assume that

$$\|\mathbf{v}_k\| = 1 \qquad \text{for} \quad 1 \leqslant j \leqslant k. \tag{9}$$

Let $\varepsilon > 0$. Then there is a positive integer N such that for every $n > N$ and every j with $1 \leqslant j \leqslant k$ we have

$$|\rho_{j,n+1} - \lambda_j \rho_{jn}| < \varepsilon \, \|\mathbf{u}_n\|. \tag{10}$$

The reason is simply that we have

$$\lim_{n \to \infty} \|\mathbf{u}_{n+1} - \mathbf{A}\mathbf{u}_n\|/\|\mathbf{u}_n\| = \lim_{n \to \infty} \|(\mathbf{A}_n - \mathbf{A})\mathbf{u}_n\|/\|\mathbf{u}_n\| = 0$$

in view of (6) and (5).

For a fixed n, let l_n be the smallest integer l with $1 \leqslant l \leqslant k$ for which $|\rho_{ln}|$ is maximal; that is,

$$|\rho_{jn}| < |\rho_{l_n n}| \geqslant |\rho_{j'n}| \qquad \text{for} \quad 1 \leqslant j < l_n < j' \leqslant k.$$

Our first claim is that, for large enough n, l_n is a nonincreasing function of n. To see this, first notice that, in view of (9), we have

$$\|\mathbf{u}_n\| \leqslant k \, |\rho_{l_n n}|$$

according to Minkowski's (triangle) inequality. Hence, by (10) we have

$$|\rho_{j,n+1} - \lambda_j \rho_{jn}| < k\varepsilon \, |\rho_{l_n,n}|. \tag{11}$$

Now let n be large enough, and fix j with $l_n < j \leqslant k$. Then

$$|\rho_{j,n+1}| < |\lambda_j| \, |\rho_{jn}| + k\varepsilon \, |\rho_{l_n,n}| < (|\lambda_{l_n}| - k\varepsilon) \, |\rho_{l_n,n}| < |\rho_{l_n,n+1}|$$

according to (11), provided ε is small enough; specifically, to ensure the validity of the second inequality, it is sufficient that

$$2k\varepsilon < \min_{1 \leqslant \nu < k} (|\lambda_\nu| - |\lambda_{\nu+1}|).$$

This implies that $l_{n+1} \leqslant l_n$, as we wanted to show.

Now, since l_n is nonincreasing for large enough n, it must be eventually constant; that is, we must have $l_n = l$ provided $n > N_0$, with some integer N_0. Notice that for $n > N_0$ we must have $\rho_{l_n} \neq 0$ since we assumed that $\mathbf{u}_n \neq 0$ for all n. As ε in (11) can be arbitrarily small provided n is large enough, it is now easy to conclude that

$$\lim_{n \to \infty} \rho_{l,n+1}/\rho_{l_n} = \lambda_l. \tag{12}$$

We are going to show that

$$\lim_{n \to \infty} \rho_{jn}/\rho_{l_n} = 0 \qquad \text{for} \quad j \neq l\,(1 \leqslant j \leqslant k). \tag{13}$$

To this end, note that we have

$$\lim_{n \to \infty} \left| \lambda_l \frac{\rho_{j,n+1}}{\rho_{l,n+1}} - \lambda_j \frac{\rho_{jn}}{\rho_{ln}} \right| = \lim_{n \to \infty} \left| \frac{\rho_{j,n+1}}{\rho_{ln}} - \lambda_j \frac{\rho_{jn}}{\rho_{ln}} \right| = 0 \qquad (1 \leqslant j \leqslant k)$$

in view of (12) and (11), as ε can be arbitrarily small in the latter equation provided n is large enough. Hence

$$\lim_{n \to \infty} \left(|\lambda_l| \left| \frac{\rho_{j,n+1}}{\rho_{l,n+1}} \right| - |\lambda_j| \left| \frac{\rho_{jn}}{\rho_{ln}} \right| \right) = 0 \qquad (1 \leqslant j \leqslant k). \tag{14}$$

Let $\langle n_\nu \rangle$ be a sequence such that

$$\lim_{\nu \to \infty} \left| \frac{\rho_{jn_\nu}}{\rho_{ln_\nu}} \right| = \limsup_{n \to \infty} \left| \frac{\rho_{jn}}{\rho_{ln}} \right| \qquad (\leqslant 1).$$

Then, using (14) with $n = n_\nu - 1$ making $\nu \to \infty$, we obtain

$$|\lambda_l| \limsup_{n \to \infty} \left| \frac{\rho_{jn}}{\rho_{ln}} \right| \leqslant |\lambda_j| \limsup_{n \to \infty} \left| \frac{\rho_{jn}}{\rho_{ln}} \right|;$$

using (14) with $n = n_\nu$ and making $\nu \to \infty$, we obtain

$$|\lambda_l| \limsup_{n \to \infty} \left| \frac{\rho_{jn}}{\rho_{ln}} \right| \geqslant |\lambda_j| \limsup_{n \to \infty} \left| \frac{\rho_{jn}}{\rho_{ln}} \right|.$$

This means

$$|\lambda_l| \limsup_{n\to\infty} \left|\frac{\rho_{jn}}{\rho_{ln}}\right| = |\lambda_j| \limsup_{n\to\infty} \left|\frac{\rho_{jn}}{\rho_{ln}}\right|.$$

If $j \neq l$ then $|\lambda_j| \neq /\lambda_l|$. As the above limit superior is not infinite (indeed, it is $\leqslant 1$), this equation is possible only if it is zero. This establishes (13); (7) with $c_n = 1/\rho_{ln}$ is an easy consequence of (13) (cf. (8)), completing the proof of the theorem. ∎

Theorem 1 easily follows from Theorem 2:

Proof of Theorem 1. Define the column vector $\mathbf{u}_n = \langle u_{0n}, ..., u_{k-1,n}\rangle^*$ (the asterisk indicates transpose) for $n > 0$ by putting

$$u_{jn} = f(n+j) \qquad (0 \leqslant j < k). \tag{15}$$

The recurrence equations

$$u_{j,n+1} = u_{j+1,n} \qquad (0 \leqslant j \leqslant k-2)$$
$$u_{k-1,n-1} = -\sum_{v=0}^{k-1} a_{vn} u_{vn}$$

for $n > 0$ are clearly equivalent to Eq. (1) (note that these equations are consistent with (15)). These recurrence equations can be written in matrix form as given in (6). In view of (2), the limit matrix $\mathbf{A}$ has the form $\mathbf{A} = [\alpha_{ij}]_{0 \leqslant i,j \leqslant k-1}$, where $\alpha_{i,i+1} = 1$ $(0 \leqslant i \leqslant k-2)$, $\alpha_{k-1,j} = -a_j$ $(0 \leqslant j \leqslant k-1)$, and all the other α_{ij} are 0; that is,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \hdotsfor{6} \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{k-2} & -a_{k-1} \end{pmatrix}.$$

The eigenvalues of this matrix are clearly the roots of Eq. (3), that is, $\zeta_1, ..., \zeta_k$. Applying Theorem 2, we obtain that either $\mathbf{u}_n = 0$ for all large enough n, in which case $f(n)$ is also 0 for all large enough n, or $\mathbf{u}_n \neq 0$ for all large enough n, in which case (7) holds. In the former case the first alternative of the conclusion of Theorem 1 is true. Assume that the latter is the case. Then (6), (7), and the definition of eigenvector imply that

$$\lim_{n\to\infty} \|\mathbf{u}_{n+1} - \zeta_l \mathbf{u}_n\| / \|\mathbf{u}_n\| = 0 \tag{16}$$

with the l given by (7). Note also that

$$\liminf_{n \to \infty} |f(n)|/\|\mathbf{u}_n\| > 0. \tag{17}$$

This holds in view of (7) (and (15) with $j=0$) since, writing v_{jl} $(0 \leqslant j \leqslant k-1)$ for the jth component of the eigenvector $\mathbf{v}_l$, the component v_{0l} is not zero. Indeed, these components satisfy the equations

$$\zeta_l v_{jl} = v_{j+1,l} \qquad (0 \leqslant j \leqslant k-2)$$

according to the eigenvector equation $\mathbf{A}\mathbf{v}_l = \zeta_l \mathbf{v}_l$. Thus if v_{0l} were 0, we would have $\mathbf{v}_l = 0$. Inequality (17) implies that $f(n) \neq 0$ for all large enough n. Therefore (15), (16), and (17) imply (4). This completes the proof of Theorem 1. ∎

After this paper was completed, we received the dissertation of R. J. Kooman [2], where he considers results similar to Theorem 2 with weaker restrictions on the eigenvalues of the limit matrix. He also discusses various results about the speed of convergence of the solutions of recurrence equations of this type.

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